Intuitionistic Fuzzy Hv-subgroups

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Abstract— *In this paper we introduce the concept of intuitionistic fuzzy Hv-subgroup and prove some related results. We also consider the fundamental relation* β^* *defined on Hv-group H and for an intuitonistic fuzzy subset* $A=\{\mu_A, \lambda_A\}$ *of H, we define an intuitonistic fuzzy subset* A_{β^*} *of* H/β^* *and we prove a theorem concerning the fundamental relation* β^*.

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I. INTRODUCTION

The concept of hyperstructure was introduced in 1934 by Marty [10]. Hyperstructures have many applications to several branches of pure and applied sciences. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Vougiouklis [26] introduced a new class of hyperstructures, the so-called H_v-structures, in which equality is replaced by non-empty intersection.

After the introduction of fuzzy sets by Zadeh [18], there have been a number of generalizations came in existence of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [17] is one of them.

At the beginning according to Atanassov the term intuitionistic means that the sum of the degree of membership and the degree of non-membership is less than 1 but after some time this condition is extended and supposed to satisfy the constraint that the sum of the degree of membership and the degree of non-membership is less than or equal to 1.

Basically the algebraic structure of intuitionistic fuzzy set is one of the interval-valued set not the intuitionistic logic [31]. In the same time the idea came in existence that the interval-valued sets are mathematically redundant up-to which level so by discussion [32] it is mathematically redundant up-to every level like power set level, fiber level, and categorical level. This naturally leads to interval-valued sets in a first step of departure away from standard fuzzy set. Indeed this is used in a long tradition in the field of Economics, Engineering and Science, etc. where the intervals were used to represent values of quantities in case of uncertainty. Currently it is studied in various domains of Information Technology, including preference modeling, learning and reasoning [33-34].

In 1971 the concept of fuzzy subgroup was introduced by Rosenfeld [1]. After that B. Davvaz [4] has given fuzzy H_v-submodules and also he [6] has given redefined fuzzy H_v-submodules and many valued implications. Zhan [12-15] et al. has given some results on fuzzy hypermodules. After that Violeta Leoreanu fotea [28] introduced fuzzy hypermodules. Zhan et al. [29-30] has given some results on L-fuzzy hypermodules. After some year M. Asghari Larimi [20-21] has given homomorphism of intuitionistic (α, β)-fuzzy H_v-submodules of H_v-modules. In 2012 Zhan et al. [16] has given intuitionistic (S, T) fuzzy H_v-submodules of H_v-modules. Recently in 2013 M. Aliakbarnia et al. [19] has given fuzzy isomorphism theorem of hyper-near modules. This paper continues this line of research on intuitionistic fuzzy H_v-subgroups.

Here it is very important to note that Intuitionistic fuzzy set is to be combined with the study of hyperstructures for more generalization of the generalized concept. In this paper, we generalize the concept of fuzzy H_v-groups [3] by using the notion of intuitionistic fuzzy set and prove some results in this respect. We also consider the fundamental relation β^* defined on H_v-group H and for an intuitonistic fuzzy subset $A=\{\mu_A, \lambda_A\}$ of H, we define an intuitonistic fuzzy subset A_{β^*} of H/β^* and prove a fundamental theorem concerning the group H/β^*.

Throughout this paper left reproduction axiom for the hypergroups is verified.

II. BASIC DEFINITIONS

In this section first we give some basic definitions for proving the further results.

Definition 2.1[7] Let X be a non-empty set. A mapping $\mu: X \to [0,1]$ is called a fuzzy set in X.

Definition 2.2[7] An intuitionistic fuzzy set A in a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denote the degree of membership and degree of non membership of each element $x \in X$ to the set A respectively and $0 \le \mu_A(x) + \lambda_A(x) \le 1$ for all $x \in X$. We shall use the symbol $A = \{\mu_A, \lambda_A\}$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$.

Definition 2.3[27] Let H be a non–empty set and $* : H \times H \to \wp^*(H)$ be a hyperoperation, where $\wp^*(H)$ is the set of all the non-empty subsets of H.

The $*$ is called weak associative if $(x*y)*z \cap x*(y*z) \neq \phi, \quad \forall x, y, z \in H$.

Where $A*B = \bigcup_{a \in A,\, b \in B} a*b, \quad \forall A, B \subseteq H.$

The $*$ is called weak commutative if $x*y \cap y*x \neq \phi, \quad \forall x, y \in H$.

$(H, *)$ is called an H_v–group if

(i) $*$ is weak associative.

(ii) $a*H = H*a = H, \quad \forall a \in H$ (Reproduction axiom).

Definition 2.4[3] Let H be a hypergroup (or H_v-group) and let μ be a fuzzy subset of H. Then μ is said to be a fuzzy subhypergroup (or fuzzy H_v-subgroup) of H if the following axioms hold:

$$(i) \min\{\mu(x), \mu(y)\} \le \inf_{\alpha \in x*y} \{\mu(\alpha)\}, \quad \forall x, y \in H.$$

(ii) For all $x, a \in H$ there exists $y \in H$ such that $x \in a*y$ and $\min\{\mu(a), \mu(x)\} \le \{\mu(y)\}$.

III. INTUITIONISTIC FUZZY HV-SUBGROUP

In this section we define intuitionistic fuzzy H_v-subgroup of a hypergroup and then we obtain the relation between a intuitionistic fuzzy subhypergroup and level subhypergroup. This relation is expressed in terms of a necessary and sufficient condition.

Definition 3.1 Let H be a hypergroup (or H_v-group). An intuitionistic fuzzy set $A = \{\mu_A, \lambda_A\}$ of H is called intuitionistic fuzzy subhypergroup (or intuitionistic fuzzy H_v-subgroup) of H if the following axioms hold:

$$(i) \min\{\mu(x), \mu(y)\} \le \inf_{\alpha \in x*y} \{\mu(\alpha)\}, \quad \forall x, y \in H.$$

(ii) For all $x, a \in H$ there exists $y \in H$ such that $x \in a*y$ and $\min\{\mu(a), \mu(x)\} \le \{\mu(y)\}$.

$$(iii) \sup_{\alpha \in x*y} \{\lambda_A(\alpha)\} \le \max\{\lambda_A(x), \lambda_A(y)\}, \quad \forall x, y \in H.$$

(*iv*) For all $x, a \in H$ there exists $y \in H$ such that $x \in a*y$ and $\{\lambda_A(y)\} \le \max\{\lambda_A(a), \lambda_A(x)\}$.

Proposition 3.2 Let $(H, \cdot)$ be a group and $A = \{\mu_A, \lambda_A\}$ be an intuitionistic fuzzy subgroup of H. If we define the following hyperoperation on H; $* : H \times H \to \wp^*(H)$, $x*y = \{t = \mu(t) = \mu(x \cdot y)\}$ then $(H, *)$ is an H_v-group and $A = \{\mu_A, \lambda_A\}$ is an intuitionistic fuzzy H_v-subgroup of H.

Proof. For all x, y, z in H we have $x \cdot (y \cdot z) \in x*(y*z)$ and $(x \cdot y) \cdot z \in (x*y)*z$. Since $(H, \cdot)$ is associative, therefore $(H, *)$ is weak associative, because $(x*y)*z \cap x*(y*z) \neq \phi$, $\forall x, y, z \in H$ and the left reproduction axiom is satisfied, i.e. for all $a \in H$, $a*H = \bigcup_{y \in H} a*y = \bigcup_{y \in H} \{z : \mu(z) = \mu(a \cdot y)\} = H$.

Now we prove that $A = \{\mu_A, \lambda_A\}$ is an intuitionistic fuzzy H_v-subgroup.

Here the conditions (i) and (ii) of definition 3.1 can be easily proved by taking $\mu = \mu_A$ in [3].

(iii)

$$\sup_{\alpha \in x*y} \{\lambda_A(\alpha)\} = \sup_{\lambda_A(\alpha) \in \lambda_A(x \cdot y)} \{\lambda_A(\alpha)\} = \lambda_A(x \cdot y) \le \max\{\lambda_A(x), \lambda_A(y)\}, \quad \forall x, y \in H$$

.*(iv)* $\forall x, a \in H, \quad x = (a \cdot a^{-1}) \cdot x = a \cdot (a^{-1} \cdot x)$, hence $x \in a*(a^{-1} \cdot x)$. Therefore it is enough to put $y = a^{-1} \cdot x$ and in this case

$$\lambda_A(y) = \lambda_A(a^{-1} \cdot x) \le \max\{\lambda_A(a^{-1}), \lambda_A(x)\} = \max\{\lambda_A(a), \lambda_A(x)\} \cdot$$

Now suppose that H is a set and $A=\{\mu_A, \lambda_A\}$ is an intuitionistic fuzzy subset of H. We define the hyperoperation $*: H\times H \to \wp^*(H)$ as follows:

Let $x, y\in H$ if $\mu_A(x)\}\le \mu_A(y)$, then $y*x = x*y = \{t : t\in H, \mu_A(x)\le \mu_A(t)\le \mu_A(y)\}$.

We will prove the following proposition.

Preposition 3.3 Let $(H,\cdot)$ be a group and $A=\{\mu_A, \lambda_A\}$ be an intuitionistic fuzzy subgroup of H, then $A=\{\mu_A, \lambda_A\}$ is an intuitionistic fuzzy H$_v$- subgroup of H .**Proof**. In order to prove the proposition, it is sufficient to prove the conditions (iii) and (iv) of definition 3.1.

(iii) Since $\forall x, y\in H$ we have

$$\sup_{\alpha\in x*y}\{\lambda_A(\alpha)\} = \sup_{\lambda_A(x)\ge\lambda_A(\alpha)\ge\lambda_A(y)}\{\lambda_A(\alpha)\} = \lambda_A(x) \le \max\{\lambda_A(x), \lambda_A(y)\}$$

.(iv) Now suppose $x, a\in H$, if $\lambda_A(\alpha)\}\ge \lambda_A(x)$, then $\lambda_A(\alpha)\}\ge \lambda_A(x)\ge \lambda_A(x)$ which implies $x\in a*x$ and if $\lambda_A(x)\}\ge \lambda_A(a)$, then $\lambda_A(x)\}\ge \lambda_A(x)\ge \lambda_A(a)$ implying $x\in x*a = a*x$, therefore if we put $y = x$, then in any case $\max\{\lambda_A(x), \lambda_A(a)\}\ge \lambda_A(y)$.

Now let H be a set and $A=\{\mu_A, \lambda_A\}$ be an intuitionistic fuzzy subset of H. Then the set $A_t = \{x\in H : A(x)\ge t\}$ is called the level set of A. We will prove the following results for H$_v$-subgroups because they are more general than hypergroups.

Theorem 3.4 Let H be an H$_v$-group and $A=\{\mu_A, \lambda_A\}$ be an intuitionistic fuzzy subset of H. Then $A=\{\mu_A, \lambda_A\}$ is an intuitionistic fuzzy H$_v$-subgroup of H if and only if for every t, $0\le t\le 1$, $A_t \ne \phi$ is an H$_v$-subgroup of H.

Proof. Let $A=\{\mu_A, \lambda_A\}$ be an intuitionistic fuzzy H$_v$-subgroup of H. $\forall x, y\in A_t$ we have $\min\{\mu_A(x), \mu_A(y)\}\ge t$ and so $\inf_{\alpha\in x\cdot y}\{\mu_A(\alpha)\}\ge t$. Therefore for every $\alpha\in x.y$ we have $\alpha\in A_t$ so $x\cdot y\subseteq A_t$. Hence for every $a\in A_t$ we have $a\cdot A_t\subseteq A_t$. Now let $x\in A_t$, then there exists $y\in H$ such that $x\in a\cdot y$ and $\min\{\mu_A(a), \mu_A(x)\}\le \mu_A(y)$. From $x\in A_t$ and $a\in A_t$ we get $\min\{\mu_A(x), \mu_A(a)\}\ge t$ and so $y\in A_t$, and this proves $A_t\subseteq a\cdot A_t$.

Conversely, assume that $\forall t,\ 0\le t\le 1,\ A_t\ne\phi$ is an H$_v$-subgroup of H.

(i) $\forall x, y\in H$, we can write $\mu_A(x)\ge \min\{\mu_A(x), \mu_A(y)\}$ and $\mu_A(y)\ge \min\{\mu_A(x), \mu_A(y)\}$ and if we put $t_0 = \min\{\mu_A(x), \mu_A(y)\}$, then $x\in A_{t_0}$ and $y\in A_{t_0}$, so $x\cdot y\subseteq A_{t_0}$. Therefore for every $\alpha\in x\cdot y$ we have $\mu_A(\alpha)\ge t_0$ implying

$$\inf_{\alpha\in x\cdot y}\{\mu_A(\alpha)\}\ge \min\{\mu_A(x), \mu_A(y)\}.$$

(ii) If $\forall a, x\in H$ we put $t_1 = \min\{\mu_A(a), \mu_A(x)\}$ then $x\in A_{t_1}$ and $a\in A_{t_1}$, so there exists $y\in A_{t_1}$, such that $x\in a\cdot y$. On the other hand since $y\in A_{t_1}$, then $t_1\le \mu_A(y)$ and hence $\min\{\mu_A(a), \mu_A(x)\}\le \mu_A(y)$.

(iii) $\forall x, y\in H$, we can write $\lambda_A(x)\le \max\{\lambda_A(x), \lambda_A(y)\}$ and $\lambda_A(y)\le \max\{\lambda_A(x), \lambda_A(y)\}$ and if we put $t_0 = \max\{\lambda_A(x), \lambda_A(y)\}$, then $x\in A_{t_0}$ and $y\in A_{t_0}$, so $x\cdot y\subseteq A_{t_0}$. Therefore for every $\alpha\in x\cdot y$ we have $\lambda_A(\alpha)\le t_0$ implying

$$\sup_{\alpha\in x\cdot y}\{\lambda_A(\alpha)\}\le \max\{\lambda_A(x), \lambda_A(y)\}.$$

(iv) If $\forall a, x\in H$ we put $t_1 = \max\{\lambda_A(a), \lambda_A(x)\}$ then $x\in A_{t_1}$ and $a\in A_{t_1}$, so there exists $y\in A_{t_1}$, such that $x\in a\cdot y$. On the other hand since $y\in A_{t_1}$, then $t_1\ge \lambda_A(y)$ and hence $\max\{\lambda_A(a), \lambda_A(x)\}\ge \lambda_A(y)$.

We can obtain the following two corollaries from Theorem 3.4.

Corollary 3.5 Let $(H,\cdot)$ be an H$_v$-group and $A=\{\mu_A, \lambda_A\}$ be an intuitionistic fuzzy H$_v$-subgroup of

H. If $0 \le t_1 \le t_2 \le 1$, then $\mu_{t_1} = \mu_{t_2}$ if and only if there is no x in H such that $t_1 \le \mu(x) \le t_2$.

Corollary 3.6 Let $(H,\cdot)$ be an H_v-group and $A=\{\mu_A, \lambda_A\}$ be an intuitionistic fuzzy H_v-subgroup of H. If the range of $A=\{\mu_A,\lambda_A\}$ is the finite set $\{t_1, t_2, ..., t_n\}$, then the set $\{A_{t_i} : 1 \le i \le n\}$ contains all the level H_v-subgroups of $A=\{\mu_A, \lambda_A\}$. Moreover if $t_1 > t_2 > ...... > t_n$, then all the level H_v-subgroups A_{t_i} form the following chain $A_{t_1} \underset{\neq}{\subset} A_{t_2} \underset{\neq}{\subset} .... \underset{\neq}{\subset} A_{t_n}$.

Theorem 3.7 Let $(H,\cdot)$ be an H_v-group. Then every H_v-subgroup of H is a level H_v-subgroup of an intuitionistic fuzzy H_v-subgroup of H.

Proof. Let A be an H_v-subgroup of H. For a fixed real number c, $0 < c \le 1$, the intuitionistic fuzzy subset A is defined as follows:

$$A(x) = \begin{cases} c, & x \in A \\ 0, & x \notin A \end{cases}$$

We have $A = A_c$ and by theorem 3.4, it is adequate to prove that A is an intuitionistic fuzzy H_v-subgroup. This is simple and we leave out for readers.

Corollary 3.8 Let $(H,\cdot)$ be an H_v-group and A be a non-empty subset of H. Then a necessary and sufficient condition for A to be an H_v-subgroup is that $A = A_{t_0}$, where A is an intuitionistic fuzzy H_v-subgroup and $0 < t_0 \le 1$.

Proof. This is obvious from Theorems 3.4 and 3.7.

Definition 3.9 Let $(H,\cdot)$ be an H_v-group and A be an intuitionistic fuzzy H_v-subgroup of H. A is called right fuzzy closed with respect to H if $\forall a,b \in H$ all the x in $b \in a \cdot x$ satisfy $\min\{A(b), A(a)\} \le A(x)$. We call A left fuzzy closed with respect to H if $\forall a,b \in H$ all the y in $b \in y \cdot a$ satisfy $\min\{A(b), A(a)\} \le A(y)$. If A is left and right fuzzy closed, then A is called fuzzy closed.

Theorem 3.10 If the intuitionistic fuzzy H_v-subgroup $A=\{\mu_A, \lambda_A\}$ is right fuzzy closed, then $A_t \cdot (H - A_t) = H - A_t$.

Proof. If $b \in A_t \cdot (H - A_t)$, then there exists $a \in A_t$ and $x \in H - A_t$ such that $b \in a \cdot x$. Therefore $A(x) < t \le A(a)$ and since A is right fuzzy closed we get $\min\{A(a), A(b)\} \le A(x)$. Hence $A(b) \le A(x) < t$ which implies $b \in H - A_t$. So we have proved $A_t \cdot (H - A_t) \subseteq H - A_t$.

On the other hand if $x \in H - A_t$, then for every $a \in A_t$ by the reproduction axiom there exists $y \in H$ such that $x \in a \cdot y$ and so it is enough to prove $y \in H - A_t$. Since A is an intuitionistic fuzzy H_v-subgroup of H, by definition we have $\min\{A(a), A(y)\} \le \inf_{\alpha \in a \cdot y} \{A(\alpha)\}$ which implies $(i) \min\{A(a), A(y)\} \le A(x)$. Since A is right fuzzy closed so $(ii) \min\{A(x), A(a)\} \le A(y)$. Now from $x \in H - A_t$ we get $a \in A_t$ and so $A(x) < t \le A(a)$. Using (ii) we obtain $A(x) \le A(y)$. Therefore $A(x) \le \min\{A(a), A(y)\}$ and by (i) the relation $\min\{A(a), A(y)\} = A(x)$ is obtained. But $A(x) < A(a)$ and hence $\min\{A(a), A(y)\} = A(y)$. So $A(x) = A(y)$ and since $x \in H - A_t$ we get $y \in H - A_t$ and the theorem is proved.

IV. THE FUNDAMENTAL RELATION

In this section we will prove a theorem concerning the fundamental relation β^*. Let $(H,\cdot)$ be an H_v-group. The relation β^* is the smallest equivalence relation on H such that the quotient H/β^* is a group. β^* is called the fundamental equivalence relation on H. This relation is studied by Corsini [22] concerning hypergroups, see also [9, 24, 27]. According to [27] if U denotes the set of all the finite products of elements of H, then a relation β can be defined on H whose transitive closure is the fundamental relation β^*. The relation β is as follows: for x and y in H we write $x\beta y$ if and only if $\{x, y\} \subseteq u$, for some $u \in U$.

Suppose β^* (a) is the equivalence class containing $a \in H$. Then the product $\otimes$ on H/β^*, the set of all the equivalence classes, is defined as follows:

$\beta^*(a) \otimes \beta^*(b) = \{\beta^*(c) : c \in \beta^*(a) \cdot \beta^*(b)\}, \quad \forall a, b \in H$. It is proved in [27] that $\beta^*(a) \otimes \beta^*(b)$ is the singleton $\{\beta^*(c)\}$ for all $c \in \beta^*(a) \cdot \beta^*(b)$. In this way H/β^* becomes a hypergroup. If we put $\beta^*(a) \otimes \beta^*(b) = \beta^*(c)$, then H/β^* becomes a group.

Definition 4.1 Let $(H, \cdot)$ be an H_v-group and $A = \{\mu_A, \lambda_A\}$ be an intuitionistic fuzzy subset of H. The intuitionistic fuzzy subset A_{β^*} on H/β^* is defined as follows:

$$\mu_{\beta^*} : H/\beta^* \to [0,1]$$
$$\lambda_{\beta^*} : H/\beta^* \to [0,1]$$
$$\mu_{\beta^*}(\beta^*(x)) = \sup_{a \in \beta^*(x)} \{\mu(a)\}$$
$$\lambda_{\beta^*}(\beta^*(x)) = \inf_{a \in \beta^*(x)} \{\lambda(a)\}$$

The concept of T-norm has been studied in [8], and definition of T-fuzzy subgroups of a group G has been introduced in [11]. Now we define T- intuitionistic fuzzy H_v-subgroup as follows:

Definition 4.2 Let $(H, \cdot)$ be an H_v-group and let $A = \{\mu_A, \lambda_A\}$ be an intuitonistic fuzzy subset of H. Then A is said to be a T-intuitionistic fuzzy H_v-subgroup of H with respect to T-norm T if the following axioms hold:

$(i)\ T(\mu_A(x), \mu_A(y)) \le \inf_{\alpha \in x \cdot y} \{\mu_A(\alpha)\}, \quad \forall x, y \in H.$

$(ii)\ \forall x, a \in H$ there exists $y \in H$ such that $x \in a \cdot y$ and $T(\mu_A(a), \mu_A(x)) \le \mu_A(y)$.

$(iii)\ T(\lambda_A(x), \lambda_A(y)) \ge \sup_{\alpha \in x \cdot y} \{\lambda_A(\alpha)\}, \quad \forall x, y \in H.$

$(iv)\ \forall x, a \in H$ there exists $y \in H$ such that $x \in a \cdot y$ and $T(\lambda_A(a), \lambda_A(x)) \ge \lambda_A(y)$.

Now we give a more general proof of the following theorem by using the concept of T-norm.

Theorem 4.3 Let T be a continuous T-norm and $A = \{\mu_A, \lambda_A\}$ be a T-intuitionistic fuzzy H_v-subgroup of H. Considering H/β^* as a hypergroup, then A_{β^*} is a T-intuitionistic fuzzy H_v-subgroup of H/β^*.

Proof. If $\mu = \mu_A$ then the conditions (i) and (ii) of definition 4.2 can be easily proved by [3].

(iii) Let $\beta^*(x)$ and $\beta^*(y)$ be two elements of H/β^*.

We can write:

$$T(\lambda_{\beta^*}(\beta^*(x), \lambda_{\beta^*}(\beta^*(y))$$
$$= T(\inf_{a \in \beta^*(x)} \{\lambda_A(a)\}, \inf_{b \in \beta^*(y)} \{\lambda_A(b)\})$$
$$= \inf_{\substack{a \in \beta^*(x) \\ b \in \beta^*(y)}} \{T(\lambda_A(a), \lambda_A(b)\} \ge \inf_{\substack{a \in \beta^*(x) \\ b \in \beta^*(y)}} \{\sup_{\alpha \in a \cdot b} \{\lambda_A(a)\}\}$$
$$\ge \inf_{\substack{a \in \beta^*(x) \\ b \in \beta^*(y)}} \{\inf_{\alpha \in a \cdot b} \{\lambda_A(a)\}\} \ge \inf_{\substack{a \in \beta^*(x) \\ b \in \beta^*(y)}} \{\inf_{\alpha \in \beta^*(a \cdot b)} \{\lambda_A(a)\}\}$$
$$= \inf_{\substack{a \in \beta^*(x) \\ b \in \beta^*(y)}} \{\lambda_{\beta^*}(\beta^*(a \cdot b))\}$$
$$= \lambda_{\beta^*}(\beta^*(a \cdot b)) = \lambda_{\beta^*}(\beta^*(a) \otimes \beta^*(b))$$

(iv) Now suppose $\beta^*(x)$ and $\beta^*(a)$ are two arbitrary elements of H/β^*. Since $A = \{\mu_A, \lambda_A\}$ is a T-intuitionistic fuzzy H_v-subgroup of H, it follows that for all $r \in \beta^*(a), s \in \beta^*(x)$ there exists $y_{r,s} \in H$ such that $r \in s \cdot y_{r,s}$ and $T(\lambda(r), \lambda(s)) \ge \lambda(y_{r,s})$. From $r \in s \cdot y_{r,s}$ it follows that $\beta^*(s) \otimes \beta^*(y_{r,s}) = \{\beta^*(r)\}$

which implies $\beta^*(x)\otimes\beta^*(y_{r,s})=\{\beta^*(a)\}$. Now if $r_1\in\beta^*(a)$ and $s_1\in\beta^*(x)$, then there exists $y_{r_1,s_1}\in H$ such that $\beta^*(s_1)\otimes\beta^*(y_{r_1,s_1})=\{\beta^*(r_1)\}$ and since $\beta^*(r_1)=\beta^*(r)$ we get $\beta^*(s_1)\otimes\beta^*(y_{r_1,s_1})=\beta^*(s)\otimes\beta^*(y_{r,s})$ and therefore $\beta^*(y_{r,s})=\beta^*(y_{r_1,s_1})$. So all the $y_{r,s}$ satisfying $T(\lambda(r),\lambda(s))\geq\lambda(y_{r,s})$ belong to the same equivalence class. Now we have:

$$\begin{aligned}&T(\lambda_{\beta^*}(\beta^*(a),\lambda_{\beta^*}(\beta^*(x))\\&=T(\inf_{r\in\beta^*(a)}\{\lambda_A(r)\},\inf_{s\in\beta^*(x)}\{\lambda_A(s)\})\\&=\inf_{\substack{r\in\beta^*(a)\\s\in\beta^*(x)}}\{T(\lambda_A(r),\lambda_A(s)\}\geq\inf_{\substack{r\in\beta^*(a)\\s\in\beta^*(x)}}\{\lambda_A(y_{r,s})\}\\&\geq\inf_{y\in\beta^*(y_{r,s})}\{\lambda_A(y)\}=\lambda_{\beta^*}(\beta^*(y_{r,s}))\end{aligned}$$

Corollary 4.4 Let $A=\{\mu_A,\lambda_A\}$ be an intuitionistic fuzzy H_v-subgroup of H. Considering H/β^* as a hypergroup, then A_{β^*} is an intuitionistic fuzzy H_v-subgroup of H/β^*.

Proof. This is obvious from Theorem 4.3, because minimum function is a continuous T-norm.

Theorem 4.5 Let H be an H_v-group and $A=\{\mu_A,\lambda_A\}$ be an intuitionistic fuzzy H_v-subgroup of H. Then A_{β^*} is an inuitionistic fuzzy subgroup of H/β^*.

Proof. Since $A=\{\mu_A,\lambda_A\}$ is an intuitionistic fuzzy H_v-subgroup, by Corollary 4.4, the first and second conditions of Definition 3.1 are satisfied, therefore

$(i)\ \min\{\mu_{\beta^*}(\beta^*(x),\mu_{\beta^*}(\beta^*(y))\leq\inf_{\beta^*(\alpha)\in\beta^*(x)\otimes\beta^*(y)}\{\mu_{\beta^*}(\beta^*(\alpha))\},\ \ \forall\beta^*(x),\beta^*(y)\in H/\beta^*$.

$(ii)\ \forall\beta^*(x),\beta^*(a)\in H/\beta^*$ there exists $\beta^*(y)\in H/\beta^*$ such that $\beta^*(x)=\beta^*(a)\otimes\beta^*(y)$ and $\min\{\mu_{\beta^*}(\beta^*(x),\mu_{\beta^*}(\beta^*(a))\}\leq\mu_{\beta^*}(\beta^*(y))$.

$(iii)\ \max\{\lambda_{\beta^*}(\beta^*(x),\lambda_{\beta^*}(\beta^*(y))\geq\sup_{\beta^*(\alpha)\in\beta^*(x)\otimes\beta^*(y)}\{\lambda_{\beta^*}(\beta^*(\alpha))\},\ \ \forall\beta^*(x),\beta^*(y)\in H/\beta^*$.

$(iv)\ \forall\beta^*(x),\beta^*(a)\in H/\beta^*$ there exists $\beta^*(y)\in H/\beta^*$ such that $\beta^*(x)=\beta^*(a)\otimes\beta^*(y)$ and $\max\{\lambda_{\beta^*}(\beta^*(x),\lambda_{\beta^*}(\beta^*(a))\}\leq\lambda_{\beta^*}(\beta^*(y))$.

Now for all $\beta^*(x)$ in H/β^* we prove that $\mu_{\beta^*}(\beta^*(x))\leq\mu_{\beta^*}(\beta^*(x)^{-1})$. Since $\beta^*(x)\in H/\beta^*$, by considering $\beta^*(a)=\beta^*(x)$ which is obtained from the second condition there exists $\beta^*(y_1)$ in H/β^* such that $\beta^*(x)=\beta^*(x)\otimes\beta^*(y_1)$ and $\min\{\mu_{\beta^*}(\beta^*(x),\mu_{\beta^*}(\beta^*(x))\}\leq\mu_{\beta^*}(\beta^*(y_1))$. From $\beta^*(x)=\beta^*(x)\otimes\beta^*(y_1)$ we obtain $\omega_H=\beta^*(y_1)$, where ω_H denotes the unit of the group H/β^*.

Therefore, we get (I) $\mu_{\beta^*}(\beta^*(x))\leq\mu_{\beta^*}(\omega_H)$.

Now considering $\beta^*(x)$, ω_H in H/β^*, by condition (ii) above there exists $\beta^*(y_2)$ in H/β^* such that $\omega_H=\beta^*(x)\otimes\beta^*(y_2)$ and $\min\{\mu_{\beta^*}(\omega_H),\mu_{\beta^*}(\beta^*(x))\}\leq\mu_{\beta^*}(\beta^*(y_2))$. From $\omega_H=\beta^*(x)\otimes\beta^*(y_2)$ we obtain $\beta^*(y_2)=\beta^*(x)^{-1}$, so we get

(II) $\min\{\mu_{\beta^*}(\omega_H),\mu_{\beta^*}(\beta^*(x))\}\leq\mu_{\beta^*}(\beta^*(x)^{-1})$

By (I) and (II) the inequality $\mu_{\beta^*}(\beta^*(x))\leq\mu_{\beta^*}(\beta^*(x)^{-1})$ is obtained.

Now for all $\beta^*(x)$ in $H/_{\beta^*}$ we prove that $\lambda_{\beta^*}(\beta^*(x)) \le \lambda_{\beta^*}(\beta^*(x)^{-1})$. Since $\beta^*(x) \in H/_{\beta^*}$ by considering $\beta^*(a) = \beta^*(x)$ which is obtained from the second condition there exists $\beta^*(y_1)$ in $H/_{\beta^*}$ such that $\beta^*(x) = \beta^*(x) \otimes \beta^*(y_1)$ and

$$\max\{\lambda_{\beta^*}(\beta^*(x), \lambda_{\beta^*}(\beta^*(x))\} \ge \lambda_{\beta^*}(\beta^*(y_1))$$

. From $\beta^*(x) = \beta^*(x) \otimes \beta^*(y_1)$ we obtain $\omega_H = \beta^*(y_1)$, where ω_H denotes the unit of the group $H/_{\beta^*}$. Therefore, we get (III) $\lambda_{\beta^*}(\beta^*(x)) \ge \lambda_{\beta^*}(\omega_H)$. Now considering $\beta^*(x)$, ω_H in $H/_{\beta^*}$, by condition (iv) above there exists $\beta^*(y_2)$ in $H/_{\beta^*}$ such that $\omega_H = \beta^*(x) \otimes \beta^*(y_2)$ and $\max\{\lambda_{\beta^*}(\omega_H), \lambda_{\beta^*}(\beta^*(x))\} \ge \lambda_{\beta^*}(\beta^*(y_2))$. From $\omega_H = \beta^*(x) \otimes \beta^*(y_2)$ we obtain $\beta^*(y_2) = \beta^*(x)^{-1}$, so we get

(IV) $\max\{\lambda_{\beta^*}(\omega_H), \lambda_{\beta^*}(\beta^*(x))\} \ge \lambda_{\beta^*}(\beta^*(x)^{-1})$

By (III) and (IV) the inequality $\lambda_{\beta^*}(\beta^*(x)) \ge \lambda_{\beta^*}(\beta^*(x)^{-1})$ is obtained.

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